Dirac's Relativistic Eq^n

This eq^n is similar to the Schroedinger eq^n for non-relativistic particles, given by Dirac in 1962.

This eq^n gives

(i) The correct interpretation of the negative energy states.

(ii) It includes the electron spin concept.

Schroedinger put

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

But could not get proper eq^n which can solve H atom properly. So he was disappointed and put this eq^n aside.

K-G put $E^2 = p^2c^2 + m^2c^4$. This eq^n solves the exact problem of H atom but according to it the density or probability density or probability amplitude comes out to be negative which is not possible.

Hence, Dirac wants to construct a eq^n in which the probability comes out to be automatically positive.

Dirac concluded that the -ve sign was due to the square of E.

Dirac wants to develop an eq^n which is of I-order in time only i.e; it is of E only not E^2.

For this he write energy in first order

$$E = H = c\vec{\alpha}\cdot p + \beta mc^2 \quad \text{——} (1)$$

(hypothesis)

Instead of $E = (p^2c^2 + m^2c^4)^{1/2}$ it consists of two ~~ter~~ terms mc^2 energy and $\vec{p}$ (p_x, p_y, p_z)

α and β are two matrices not the numbers because they do not commute with each other

Put this value of energy H in Schroedinger eq^n.

$$\hat{H}\Psi(r,t) = i\hbar \frac{\partial \Psi(r,t)}{\partial t} \quad \text{——} (2)$$

$$(c\vec{\alpha}\cdot p + \beta mc^2)\Psi(r,t) = i\hbar \frac{\partial \Psi(r,t)}{\partial t}$$

Substituting operator for p viz $p = -i\hbar\nabla$, we obtain

$$[c\vec{\alpha}\cdot(-i\hbar\nabla) + \beta mc^2]\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

$$\boxed{\left(i\hbar\frac{\partial}{\partial t} + i\hbar c\vec{\alpha}\cdot\nabla - \beta mc^2\right)\psi(x,t) = 0} \quad (4)$$

This is the Dirac's eqn for free particle.

Again, from (3)

$$(c\vec{\alpha}\cdot p + \beta mc^2)\psi(x,t) = E\psi(x,t)$$

or $(E - c\vec{\alpha}\cdot p - \beta mc^2)\psi = 0$ —— (5)

Operating above eqn by $(E + c\vec{\alpha}\cdot p + \beta mc^2)$ from left we get

$$(E + c\vec{\alpha}\cdot p + \beta mc^2)(E - c\vec{\alpha}\cdot p - \beta mc^2)\psi = 0$$

$$[E^2 - (c\vec{\alpha}\cdot p + \beta mc^2)^2]\psi = 0$$

$$[E^2 - c^2(\vec{\alpha}\cdot p)^2 - \beta^2 m^2 c^4 - mc^3(\vec{\alpha}\cdot p)\beta$$
$$- mc^3\beta\vec{\alpha}\cdot p]\psi = 0 \quad (6)$$

But $\vec{\alpha} = i\alpha_x + j\alpha_y + k\alpha_z$, $p = ip_x + jp_y + kp_z$

$\therefore \quad \vec{\alpha}\cdot p = \alpha_x p_x + \alpha_y p_y + \alpha_z p_z$

therefore eq^n (6) becomes

$$\Big[E^2 - c^2(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)^2 - \beta^2 m^2 c^4$$
$$- mc^3(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)\beta$$
$$- mc^3 \beta(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)\Big]\psi = 0 \quad (7)$$

or

$$\Big[E^2 - c^2\{\alpha_x^2 p_x^2 + \alpha_y^2 p_y^2 + \alpha_z^2 p_z^2$$
$$+ (\alpha_x\alpha_y + \alpha_y\alpha_x)p_x p_y +$$
$$(\alpha_y\alpha_z + \alpha_z\alpha_y)p_y p_z +$$
$$(\alpha_z\alpha_x + \alpha_x\alpha_z)p_z p_x\} - \beta^2 m^2 c^4$$
$$- mc^3\{(\alpha_x\beta + \beta\alpha_x)p_x + (\alpha_y\beta + \beta\alpha_y)p_y$$
$$+ (\alpha_z\beta + \beta\alpha_z)p_z\Big]\psi = 0 \quad (8)$$

where the substitutions $i\hbar\frac{\partial}{\partial t}$ for E and $+i\hbar\nabla$ for p are implied. Klein-Gordon eq^n is

$$\left[E^2 - c^2(p_x^2 + p_y^2 + p_z^2) - m^2c^4\right]\psi = 0 \quad \text{(9)}$$

comparing eq^n (8) and (9), we obtain

$$
\left.
\begin{aligned}
&\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1 \\
&\alpha_x\alpha_y + \alpha_y\alpha_x = 0; \\
&(\alpha_y\alpha_z + \alpha_z\alpha_y) = 0; \\
&(\alpha_z\alpha_x + \alpha_x\alpha_z) = 0 \\
&\alpha_x\beta + \beta\alpha_x = 0;\ (\alpha_y\beta + \beta\alpha_y) = 0; \\
&(\alpha_z\beta + \beta\alpha_z) = 0
\end{aligned}
\right\} \quad \text{(10)}
$$

That is the four quantities α_x, α_y, α_z and β have the following properties :-

(i) their squares are unity and
(ii) they anticommute with one another in pairs

Since $\vec{\alpha}$ and β anticommute rather than commute with each other, they cannot be numbers.

Moreover the quantities of this type can be expressed in terms of matrices and it is convenient to find a matrix representation of them.